Lambert's Algorithm for $a = \sqrt[n]{c}$, $c > 0$.

(J. H. Lambert, Beiträge zum Gebrauche der Mathematik und deren Anwendungen, Zweyter Theil, Erster Abschnitt, 1770, p. 152.)

Start with $x > 0$. Iterate

$$x \leftarrow g(x) := x\frac{(n+1)c + (n-1)x^n}{(n-1)c + (n+1)x^n}$$

Therefore <u>all iterates</u> are positive. More precisely,

$$x_{k+1} = g(x_k), \quad k = 0, 1, 2, \ldots,$$

with $x_0 = a$, initial guess. In practice, on a binary machine (most are), we can scale so that $1 \le 2c \le 2^n$. Then $x_0 = 1$ is a pretty good initial guess. Rearrange a bit:

$$\begin{aligned} g(x) - x &= x\left(\frac{(n+1)c + (n-1)x^n}{(n-1)c + (n+1)x^n} - 1\right) \\ &= 2x\frac{c - x^n}{(n-1)c + (n+1)x^n} \end{aligned}$$

Thus

$$\begin{array}{ccccccccc} 0 & < & x_0 & < & a & \Rightarrow & x_0 & < & x_1, \\ & & x_0 & > & a & \Rightarrow & x_0 & > & x_1 \end{array}$$

But we'll show a lot more.

Again,

$$g(x) - a = x - a + 2x\frac{c - x^n}{(n-1)c + (n+1)x^n}.$$

To greatly simplify matters we make the change of variables: $z = \dfrac{x}{a}$, $x = az$. Then $x = a \Leftrightarrow z = 1$ and

$$\frac{g(az)}{a} - 1 = z - 1 - 2z\frac{z^n - 1}{(n-1) + (n+1)z^n}.$$

But

$$z^n - 1 = (z-1)(1 + z + z^2 + \cdots + z^{n-1})$$

so

$$\begin{aligned}
\frac{g(az)}{a} - 1 &= (z-1)\left(1 - 2z\frac{1+z+z^2+\cdots+z^{n-1}}{(n-1)+(n+1)z^n}\right) \\
&= (z-1)\frac{(n-1)-2z-2z^2-\cdots-2z^{n-1}-2z^n+(n+1)z^n}{(n-1)+(n+1)z^n} \\
&= (z-1)\frac{(n-1)-2z-2z^2-\cdots-2z^{n-1}+(n-1)z^n}{(n-1)+(n+1)z^n} \\
&=: (z-1)\frac{p_0(z)}{q(z)}.
\end{aligned}$$

We have

$$p_0(1) = (n-1) - 2(n-1) + (n-1) = 0$$

so $z-1$ divides $p_0(z)$. Let

$$p_1(z) := \frac{p_0(z)}{z-1} =: a_0 + a_1 z + \cdots + a_{n-1}z^{n-1}.$$

Then

$$\begin{aligned}
(-1+z)/p_1(z) &= (-1+z)(a_0 + a_1 z + \cdots + a_{n-1}z^{n-1}) \\
&= -a_0 - a_1 z - a_2 z^2 - \cdots - a_{n-1}z^{n-1} \\
&\qquad + a_0 z + a_1 z^2 + \cdots + a_{n-2}z^{n-1} + a_{n-1}z^n \\
&= -a_0 + (a_0 - a_1)z + (a_1 - a_2)z^2 + \cdots + (a_{n-2} - a_{n-1})z^{n-1} + a_{n-1}z^n \\
&= (n-1) - 2z - 2z^2 - \cdots - 2z^{n-1} + (n-1)z^n = p_0(z).
\end{aligned}$$

Now, two polynomials are equal if and only if their coefficients are equal.

Check this out. Hence

$$\begin{array}{lcl}
-a_0 & = & n-1 \\
a_0 - a_1 & = & -2 \\
a_1 - a_2 & = & -2 \\
\vdots & & \\
a_{k-1} - a_k & = & -2 \\
\vdots & & \\
a_{n-2} - a_{n-1} & = & -2 \\
a_{n-1} & = & n-1
\end{array}
\quad \Rightarrow \quad
\begin{array}{lcl}
a_0 & = & -n+1 \\
a_1 & = & a_0 + 2 = -n+3 \\
a_2 & = & a_1 + 2 = -n+5 \\
\vdots & & \\
a_k & = & -m+2k+1 \\
\vdots & & \\
a_{n-1} & = & n-1
\end{array}$$

So

$$\begin{aligned}
p_1(z) &= \sum_{k=1}^{n-1} a_k z^k \\
&= \sum_{k=0}^{n-1} (-(n-1)+2k) z^k \\
&= -(n-1)\sum_{k=0}^{n-1} z^k + 2\sum_{k=1}^{n-1} k z^k
\end{aligned}$$

and

$$\begin{aligned}
p_1(1) &= -(n-1)\sum_{k=0}^{n-1} 1 + 2\sum_{k=1}^{n-1} k \\
&= -(n-1)n + 2\frac{n(n-1)}{2} = 0,
\end{aligned}$$

showing that $z-1$ divides $p_1(z)$. Let

$$p(z) := \frac{p_1(z)}{z-1} =: b_0 + b_1 z + \cdots + b_{n-2} z^{n-2}.$$

Then

$$\begin{aligned}
&(-1+z)(b_0 + b_1 z + b_2 z^2 \cdots + b_{n-2} z^{n-2}) \\
&= -b_0 - b_1 z - b_2 z^2 - \cdots - b_{n-3} z^{n-3} - b_{n-2} z^{n-2} \\
&\qquad + b_0 z + b_1 z^2 + \cdots + b_{n-4} z^{n-4} + b_{n-3} z^{n-3} + b_{n-2} z^{n-1} \\
&= p_1(z).
\end{aligned}$$

Hence

$$b_{k-1} - b_k = a_k, \qquad k = 0, 1, \ldots, n-1,$$

in which

$$b_{-1} := b_{n-1} := 0.$$

So

$$b_k = b_{k-1} - a_k, \qquad k = 0, 1, \ldots, n-2,$$

and, as a check, we should have

$$b_{n-2} = a_{n-1} = n - 1.$$

Sum to get

$$\begin{aligned}
b_0 &= \quad -a_0 \text{ since } \ b_{-1} = 0 \\
b_1 &= b_0 - a_1 = -a_0 - a_1 \\
b_2 &= b_1 - a_2 = -a_0 - a_1 - a_2 \\
&\vdots \\
b_k &= -\textstyle\sum_{j=0}^{k} a_j \\
&= -\textstyle\sum_{j=0}^{k}(-n + 2j + 1) \\
&= \textstyle\sum_{j=0}^{k}(n - 1 - 2j) \\
&= (n-1)\textstyle\sum_{j=0}^{k} 1 - 2\sum_{j=0}^{k} j \\
&= (n-1)(k+1) - 2\tfrac{k(k+1)}{2} \\
&= (n-1)(k+1) - k(k+1) \\
&= (n-k-1)(k+1).
\end{aligned}$$

In particular

$$\begin{aligned}
b_{n-2} &= (n - n + 2 - 1)(n - 2 + 1) \\
&= n - 1.
\end{aligned}$$

<u>Summary:</u>

$$p_0(z) = (z-1)p_1(z) = (z-1)^2 p(z)$$

with

$$p(z) = \sum_{k=0}^{n-2}(k+1)(n-k-1)z^k$$

and all coefficients are positive. In particular,

$$\begin{aligned}
p(1) &= n\sum_{k=1}^{n-1} k - \sum_{k=1}^{n-1} k^2 \\
&= n\frac{(n-1)n}{2} - \frac{(n-1)\left(n-\frac{1}{2}\right)n}{3} \\
&= \frac{1}{6}(n-1)n(n+1) = \frac{n(n^2-1)}{6}
\end{aligned}$$

also

$$q(1) = (n-1) + (n+1) = 2n,$$

so

$$\frac{p(1)}{q(1)} = \frac{n^2-1}{12} > 0, \qquad \text{for } \ n > 1.$$

Transforming back to x we get

$$g(x) - a = \frac{(x-a)^3}{a^2}\frac{p\left(\frac{x}{a}\right)}{q\left(\frac{x}{a}\right)},$$

that is

$$x_{k+1} = a + \frac{(x_k-a)^3}{a^2}\underbrace{\frac{p\left(\frac{x_k}{a}\right)}{q\left(\frac{x_k}{a}\right)}}_{>0}$$

so

$$x_k > a \;\Rightarrow\; x_{k+1} > a$$

and

$$x_k < a \;\Rightarrow\; x_{k+1} < a.$$

All together

$$\begin{aligned}
0 < \quad & x_0 < a \;\Rightarrow\; x_0 < x_1 < \cdots < x_n \nearrow x^* \\
& x_0 > a \;\Rightarrow\; x_0 > x_1 < \cdots > x_n \searrow x^*.
\end{aligned}$$

And, by passing to the limit in the iteration equation, because $g(x)$ is continuous for $x > 0$,

$$x^* = g(x^*) = x^* \frac{(n+1)c + (n-1)(x^*)^n}{(n-1)c + (n+1)(x^*)^n}.$$

But $x^* > 0$ so it can be cancelled to get

$$(n-1)c + (n+1)(x^*)^n = (n+1)c + (n-1)(x^*)^n,$$

that is

$$2(x^*)^n = 2c$$

or

$$(x^*)^n = c.$$

Since $x^* > 0$, $c > 0, x^*$ must be the unique positive n^{th} root of $c: \ x^* = a = \sqrt[n]{c}$. Convergence is cubic since

$$\frac{x_{k+1} - a}{(x_k - a)^3} \to \frac{1}{a^2} \frac{p(1)}{q(1)} = \frac{n^2 - 1}{12a^2}, \qquad \text{as } k \to +\infty.$$

Lambert's algorithm <u>is</u> Newton's method for

$$f(x) = x^{\frac{n+1}{2}} - \frac{c}{x^{\frac{n-1}{2}}}$$

This was confusing at first, for $n = 2$. We're trying to compute $\sqrt{c}$ but f involves square roots! Never worry, we have

$$f(x) = \frac{x^n - c}{x^{\frac{n-1}{2}}},$$

$$\begin{aligned} f'(x) &= \frac{n+1}{2} x^{\frac{n-1}{2}} + \frac{n-1}{2} \frac{c}{x^{\frac{n+1}{2}}} \\ &= \frac{1}{2} \frac{(n+1)x^n + (n-1)c}{x^{\frac{n+1}{2}}} \\ \frac{f'(x)}{f(x)} &= \tfrac{1}{2} \tfrac{(n+1)x^n + (n-1)c}{x^{\frac{n+1}{2}}} \tfrac{x^{\frac{n-1}{2}}}{x^n - c} \\ &= \frac{1}{2} \frac{(n-1)c + (n+1)x^n}{x(x^n - c)}, \end{aligned}$$

a rational function! In trying to find <u>all</u> functions so the last equation holds we observe that

$$\frac{f'(x)}{f(x)} = \frac{d}{dx} \ln f(x)$$

(when $f(x) > 0$). Hence

$$\ln f(x) = \frac{1}{2} \int \frac{(n-1)c + (n+1)x^n}{x(x^n - c)} dx$$

and we are naturally led to the integral

$$\int \frac{dz}{z(z^n - 1)}.$$

That seemed tedious, but not uninteresting, to do via partial fractions. But in fact, by the above stuff, it's

$$\int \frac{dz}{z(z^n - 1)} = \frac{1}{n} \ln \left| 1 - \frac{1}{z^n} \right| + k,$$

as can be seen by differentiating both sides (consider cases $z^n > 1$ and $z^n < 1$).

Lambert's algorithm <u>is</u> Halley's algorithm of 1694 applied to $f(x) = x^n - c$.